TRINITY COLLEGE

Mathematics Specialist Units 3 & 4
Test 4 2016

Section 1 Calculator Free

Vector Calculus and
Integration using Trigonometric Identities and Substitution

STUDENT'S NAME: SOLUTIONS

DATE: Friday 20th May **TIME:** 25 minutes **MARKS**: 27

INSTRUCTIONS:

Standard Items: Pens, pencils, pencil sharper, eraser, correction fluid/tape, ruler, highlighters, Formula Sheet.

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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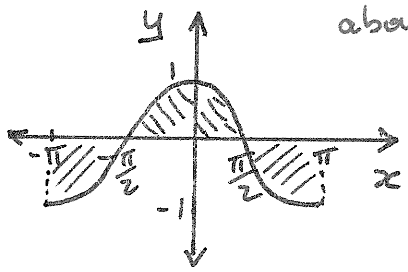
1. (5 marks)

Evaluate the following:

(a) $\int_{-\pi}^{\pi} \cos x \, dx$ [2]

$= \underline{\underline{0}}$ ✓✓

$y = \cos x$ is an even function over the interval (symmetric about y-axis)



(b) $\int_{-\pi}^{\pi} |\cos x| \, dx$ [3]

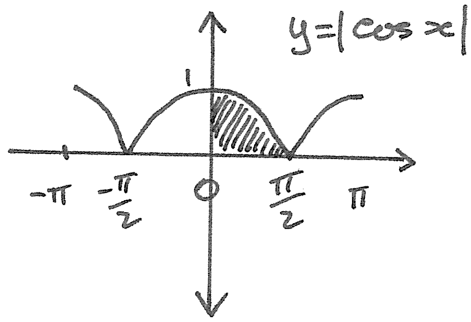
$= 4\int_0^{\frac{\pi}{2}} \cos x \, dx$ ✓

$= 4\left[\sin x\right]_0^{\frac{\pi}{2}}$ ✓

$= 4\left(\sin\frac{\pi}{2} - \sin 0\right)$

$= 4(1-0)$

$= \underline{\underline{4}}$ ✓



2. (5 marks)

Evaluate the definite integral:

$$\int_{-1}^{\frac{1}{3}} \frac{1}{x^2}\sqrt{1+\frac{1}{x}}\,dx, \qquad \text{given} \quad \int f'(x)(f(x))^n\,dx = \frac{(f(x))^{n+1}}{n+1}+c$$

Check: If $f(x) = 1+\frac{1}{x}$

$= 1 + x^{-1}$

$\Rightarrow f'(x) = -x^{-2}$

$= \frac{-1}{x^2}$

$$= -\int_{-1}^{\frac{1}{3}} -\frac{1}{x^2}\left(1+\frac{1}{x}\right)^{\frac{1}{2}} dx \quad \checkmark\checkmark$$

$$= \left[-\frac{2}{3}\left(1+\frac{1}{x}\right)^{\frac{3}{2}}\right]_{-1}^{\frac{1}{3}} \quad \checkmark$$

$$= -\frac{2}{3}(1+3)^{\frac{3}{2}} - 0 \quad \checkmark$$

$$= -\frac{2}{3}(4)^{\frac{3}{2}}$$

$$= \underline{\underline{\frac{-16}{3}}} \quad \checkmark$$

or is it?

see Q5 in section 2.

3. (9 marks)

Determine the following integrals:

(a) $\int \frac{\sin(2x)}{\cos x}\,dx$ [4]

$$= \int \frac{2\sin x \cancel{\cos x}}{\cancel{\cos x}}\,dx$$

$$= -2\cos x + c$$

(b) $\int \sin^4(2x)\,dx$ [5]

$$= \frac{1}{4}\int (1-\cos 4x)^2\,dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos 4x + \cos^2 4x\right) dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos 4x + \frac{1}{2} + \frac{1}{2}\cos 8x\right) dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos 4x + \frac{1}{2}\cos 8x\right) dx$$

$$= \frac{1}{4}\left(\frac{3x}{2} - \frac{\sin 4x}{2} + \frac{\sin 8x}{16}\right) + c$$

$$= \frac{3x}{8} - \frac{\sin 4x}{8} + \frac{\sin 8x}{64} + c$$

since $\cos(2x) = 1 - 2\sin^2 x$

$\Rightarrow 2\sin^2 x = 1 - \cos(2x)$

$\Rightarrow 2\sin^2(2x) = 1 - \cos(4x)$

$\Rightarrow \sin^2(2x) = \frac{1}{2}(1 - \cos(4x))$

$\Rightarrow \sin^4(2x) = \frac{1}{4}(1 - \cos(4x))^2$

and

$\cos 2x = 2\cos^2 x - 1$

$\Rightarrow 2\cos^2 x = 1 + \cos 2x$

$\Rightarrow \cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$\Rightarrow \cos^2 4x = \frac{1}{2}(1 + \cos 8x)$

4. (8 marks)

Determine the following using the given substitution:

(a) $\int \frac{dx}{1+x^2}$ Let $x = \tan\theta$ $\Rightarrow \frac{dx}{d\theta} = \sec^2\theta$ [4]

$\Rightarrow dx = \sec^2\theta \cdot d\theta$

$= \int \frac{\sec^2\theta}{1+\tan^2\theta}\, d\theta$

$= \int \frac{\sec^2\theta}{\sec^2\theta}\, d\theta$

$= \int 1 \, d\theta$

$= \theta + c$

$= \tan^{-1}(x) + c$

(b) $\int_0^1 \frac{1}{\sqrt{4-x^2}}\, dx$ Let $x = 2\sin\theta$ $\Rightarrow \frac{dx}{d\theta} = 2\cos\theta$ [4]

When $x = 0$

$0 = 2\sin\theta$

$\therefore \theta = 0$

$\Rightarrow dx = 2\cos\theta \cdot d\theta$

When $x = 1$

$1 = 2\sin\theta$

$\Rightarrow \sin\theta = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{6}$

$= \int_0^{\frac{\pi}{6}} \frac{1}{\sqrt{4-4\sin^2\theta}}\, 2\cos\theta \, d\theta$

$= \int_0^{\frac{\pi}{6}} \frac{1}{\cancel{\sqrt{4\cos^2\theta}}} \cdot \cancel{2\cos\theta}\, d\theta$

$= \int_0^{\pi/6} 1 \, d\theta$

$= [\theta]_0^{\frac{\pi}{6}}$

$= \frac{\pi}{6}$

End of Questions

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Section 2 Calculator Assumed

Vector Calculus and
Integration using Trigonometric Identities and Substitution

STUDENT'S NAME: SOLUTIONS

DATE: Friday 20th May **TIME:** 25 minutes **MARKS**: 27

INSTRUCTIONS:

Standard Items: Pens, pencils, pencil sharper, eraser, correction fluid/tape, ruler, highlighters, Formula Sheet retained from Section 1.

Special Items: Drawing instruments, templates, three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment).

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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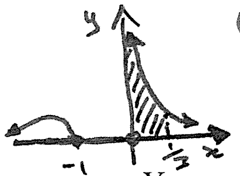
5. (4 marks)

You will recall that in Question 2 you were asked to evaluate the definite integral: ie. a Unique real number ie the limit of a (Riemann) sum.



Consider the graph of the function.

$$\int_{-1}^{\frac{1}{3}} \frac{1}{x^2}\sqrt{1+\frac{1}{x}}\,dx = \infty$$

Does not ✓ exist; asymptote at $x=0$, ✓ not defined for $-1 < x \leq 0$ ✓✓

You may well have obtained the result of $-\dfrac{16}{3}$.

Now * use your calculator to evaluate the above definite integral. Comment on your result.

Being able to antidifferentiate (to find the primitive) does not imply the definite integral will exist.

Discontinuities and Undefined values cause problems!

$\int_a^b f(x)\,dx = F(b) - F(a)$ requires f to be continuous on the interval $a < x < b$.

*Your classPad will do nothing unless you place it in Complex mode!

6. (13 marks)

The orbit of a planet around its sun is given by the position vector

$$\mathbf{r}(t)=\cos\left(\frac{\pi t}{200}\right)\mathbf{i}-2\sin\left(\frac{\pi t}{200}\right)\mathbf{j}$$

where t is time measured in Earth days and distance is in appropriate astronomical units.

(a) Determine $\mathbf{r}(0)$ and $\mathbf{r}(400)$. Hence calculate the length of the planet's year. [3]

$\underset{\sim}{r}(0) = \underset{\sim}{i}$ ✓

$\underset{\sim}{r}(400) = \underset{\sim}{i}$ ✓

$\therefore$ Length of the planet's year is 400 Earth days ✓

(b) Show that the distance of the planet from its sun is $d=\sqrt{1+3\sin^2\left(\frac{\pi t}{200}\right)}$ [2]

$$d = |\underset{\sim}{r}(t)| = \sqrt{\cos^2\left(\frac{\pi t}{200}\right) + \left(-2\sin\left(\frac{\pi t}{200}\right)\right)^2}$$ ✓

$$= \sqrt{\cos^2\left(\frac{\pi t}{200}\right) + 4\sin^2\left(\frac{\pi t}{200}\right)}$$

$$= \sqrt{\cos^2\left(\frac{\pi t}{200}\right) + \sin^2\left(\frac{\pi t}{200}\right) + 3\sin^2\left(\frac{\pi t}{200}\right)}$$

$$= \sqrt{1 + 3\sin^2\left(\frac{\pi t}{200}\right)}$$ ✓

QED.

(c) At what time during the planet's year, is it a maximum distance from its sun? [2]

Using (b), max distance when $\sin^2\left(\frac{\pi t}{200}\right) = 1$ ✓

$$\Rightarrow \frac{\pi t}{200} = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$ ✓

$\therefore$ $t = 100, 300$ days of the year.

(d) Determine the planet's orbiting speed when it is at its maximum distance from its sun. [2]

$$\underset{\sim}{v}(t) = \frac{-\pi}{200}\sin\left(\frac{\pi t}{200}\right)\underset{\sim}{i} - \frac{2\pi}{200}\cos\left(\frac{\pi t}{200}\right)\underset{\sim}{j}$$

$$\Rightarrow \underset{\sim}{v}(100) = \frac{-\pi}{200}\underset{\sim}{i}$$

$$\text{and } \underset{\sim}{v}(300) = \frac{-\pi}{200}(-1)\underset{\sim}{i}$$

$$\therefore \quad \text{speed} = |\underset{\sim}{v}(t)| = \underline{\underline{\frac{\pi}{200}}}$$ appropriate astro. units/day.

(e) Show that the acceleration vector is a scalar multiple of the position vector. [2]

$$\underset{\sim}{a}(t) = -\frac{\pi^2}{200^2}\cos\left(\frac{\pi t}{200}\right)\underset{\sim}{i} + \frac{2\pi^2}{200^2}\sin\left(\frac{\pi t}{200}\right)\underset{\sim}{j}$$

$$= \frac{-\pi^2}{200^2}\left(\cos\left(\frac{\pi t}{200}\right)\underset{\sim}{i} - 2\sin\left(\frac{\pi t}{200}\right)\underset{\sim}{j}\right)$$

$$= \frac{-\pi^2}{40000}\underset{\sim}{r}(t)$$ i.e. a scalar multiple as required.

(f) State the Cartesian equation of the path of the planet. [2]

$$x = \cos\left(\frac{\pi t}{200}\right) \qquad y = -2\sin\left(\frac{\pi t}{200}\right)$$

$$\Rightarrow -\frac{1}{2}y = \sin\left(\frac{\pi t}{200}\right)$$

$$x^2 + \left(-\frac{1}{2}y\right)^2 = 1$$ since $\cos^2\theta + \sin^2\theta = 1$

$$\Rightarrow x^2 + \frac{y^2}{4} = 1$$

$$\therefore \quad 4x^2 + y^2 = 4$$

as requested.

7. (10 marks)

$\Rightarrow c_1 = 18, \quad c_2 = 0$

An object is launched from a point with position vector $\mathbf{r}(0)=18\mathbf{i}+4\mathbf{j}$ metres. The velocity vector of the object, t seconds after projection, is given by $\mathbf{v}(t)=-\mathbf{i}-\dfrac{1}{2\sqrt{16-t}}\mathbf{j}$ ms^{-1} .

(a) Determine the position vector of the object at time t seconds. [2]

$$\underset{\sim}{r}(t) = (-t+c_1)\underset{\sim}{i} + (\sqrt{16-t} + c_2)\underset{\sim}{j} \quad ✓$$

$$= (-t+18)\underset{\sim}{i} + \sqrt{16-t}\,\underset{\sim}{j} \quad ✓$$

$$\underset{\sim}{r}(t) = \begin{pmatrix} 18-t \\ \sqrt{16-t} \end{pmatrix}$$

(b) Determine the position vector of the point where the object hits the ground. [3]

Hits the ground when $\sqrt{16-t} = 0$ ✓ Vertical component is zero.

$\Rightarrow t = 16$ ✓

$$\underset{\sim}{r}(16) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \text{ metres.} \quad ✓$$

(c) Determine the speed and the direction of the object at t = 12 seconds. [3]

$$\underset{\sim}{v}(12) = \begin{pmatrix} -1 \\ -\frac{1}{4} \end{pmatrix} \quad ✓$$

Using ClassPad:

toPol([-1, -0.25])

= 1.03 ∠ -2.90 (2d.p.)

$|\underset{\sim}{v}(12)|$

$= \sqrt{(-1)^2+(-\frac{1}{4})^2} = \underline{\underline{\dfrac{\sqrt{17}}{4} \text{ ms}^{-1}}}$ ✓ ie. $\dfrac{\sqrt{17}}{4}$ ms^{-1} ; $\underline{\underline{-166^\circ}}$. ✓

(d) Calculate the total distance travelled by the object in the first 12 seconds. [2]

$$\text{Total Dist. Travelled} = \int_0^{12} |\underset{\sim}{v}(t)|\, dt \quad ✓$$

$$(\text{Using ClassPad}) = \int_0^{12} \text{norm}\left(\left[-1, \frac{-1}{2\sqrt{16-t}}\right]\right) dt$$

$$= \underline{\underline{12.172 \text{ m}}} \ (3d.p.) \quad ✓$$

End of Questions